

Hedonic preferences, symmetric loss aversion and the willingness to pay-willingness to accept gap

Mogens Fosgerau * Bruno De Borger †

March 17, 2009

Abstract

We consider a consumer who makes choices based on choice preferences exhibiting loss aversion, which causes a gap between the willingness to pay and the willingness to accept. We show how loss aversion may be given a rational basis, providing an explicit link from hedonic to choice preferences. We then define the degree of asymmetry of loss aversion and show that the hedonic marginal rate of substitution (MRS) is identified from choices exhibiting loss aversion if the degree of asymmetry is known. Finally, we show that symmetric loss aversion is rational in the sense that it leads to maximal expected hedonic utility. It is thus possible to back out the hedonic MRS between two goods from observations of choices under loss aversion.

*Technical University of Denmark, mf@transport.dtu.dk and Centre for Transport Studies, Royal Institute of Technology, Sweden

†University of Antwerpen, bruno.deborger@ua.ac.be

Keywords: Behavioral public economics; valuation of non-market goods; loss aversion

1 Introduction

A large literature has developed on the valuation of non-market goods in a variety of sectors, including health, transportation, environmental amenities, marketing, etc. The fundamental objective is to measure the marginal rate of substitution (MRS) between money and a given non-market good under question. Various measures are available for this MRS, including the willingness to pay (WTP) and the willingness to accept (WTA). A remarkably consistent finding, however, is the large gap between these measures in valuation studies. [Horowitz and McConnell \(2003\)](#) review more than 200 valuation studies and find the mean of the ratio (WTA/WTP) to exceed 7, a difference which is hard to rationalize within a standard Hicksian framework.

Loss aversion ([Tversky and Kahneman, 1991](#); [Kahneman and Tversky, 1979](#)) provides an appealing explanation of the WTP/WTA gap. Under loss aversion, losses cause a greater loss of utility than same size gains. Within this framework, [Bateman et al. \(1997\)](#) show that WTA must necessarily exceed WTP, and they provide experimental evidence for the magnitude of the effect. Based on a large survey studying people's valuation of time, [De Borger and Fosgerau \(2008\)](#) confirm the power of loss aversion in explaining differences in valuation measures.

Even if loss aversion is able to explain the large observed differences between different valuation measures, this has raised new questions. Our main question is how to relate the various valuation measures to the MRS of interest. This is clearly important as these measures may be very different. In answering this question we postulate a link from underlying hedonic preferences to choice preferences. We take as given that the MRS of interest relates to the hedonic preferences while ob-

served choices are generated from choices preferences. As we rely on rationality arguments it becomes relevant whether loss aversion may be considered rational.

We begin in section 2 by suggesting a model that provides a link from hedonic to choice preferences in which loss aversion is rational. We use the theory of [Gilboa and Schmeidler \(1989\)](#) to motivate the formulation of reference-dependent choice utility as the minimum of expected hedonic utility where the minimum is taken over a set of probability measures that represents uncertainty. Specifying uncertainty as a function of the distance to the reference leads to an expression for reference-dependent choice utility that incorporates loss aversion and closely resembles formulations of reference-dependent choice utility that have previously appeared in the literature.

Section 3 presents four different valuation measures in the situation when reference-dependent choice utility may have kinks at the reference. In addition to the WTP and the WTA, these measures are the equivalent gain (EG) and the equivalent loss (EL), i.e. the equivalent variation for gains and losses respectively. We find that $WTP \cdot WTA = EG \cdot EL$, provided reference-dependent choice utility is left and right differentiable at the reference. Thus, if subjects behave as if maximising some left and right differentiable function, then the above equality obtains. Empirical measurements of WTP , WTA , EG and EL may then be used to check whether observed behaviour is consistent with subjects behaving as if maximising some function.

In section 4, we use the model of section 2 to follow [Munro and Sugden \(2003\)](#), [Bernheim and Rangel \(2007\)](#) and [Kőszegi and Rabin \(2006\)](#) in explicitly linking choice preferences and reference-free hedonic preferences.¹ We do so

¹This line of attack is also advocated by [Beshears et al. \(2007\)](#). See also [Kahneman and Sugden](#)

within the framework of the 'kinked at the reference'-value functions suggested by [Tversky and Kahneman \(1991\)](#). We then introduce the concept of the degree of asymmetry of loss aversion and in particular the concept of symmetric loss aversion. The degree of asymmetry measures how much loss aversion tilts the reference-dependent marginal rates of substitution (corresponding to the different valuation measures) away from the unique hedonic marginal rate of substitution. Importantly, we show that the hedonic MRS is not identified separately from the degree of asymmetry without further assumptions. On the other hand, if the degree of asymmetry is known, then the hedonic MRS can be identified from WTP and WTA or equivalently from EG and EL. So the problem of identifying the hedonic MRS is reduced to the problem of identifying the degree of asymmetry.

Since loss aversion may be considered rational, it also makes sense to consider which degree of asymmetry of loss aversion is rational. [Section 5](#) argues that symmetric loss aversion is rational in the sense that it maximizes expected hedonic utility. Moreover, using the [Munro and Sugden \(2003\)](#) concept of a reflexive optimum we show that the condition that the hedonic optimum is a reflexive optimum implies that the hedonic MRS is bounded between the WTP and the WTA. [Section 6](#) concludes.

2 A rational basis for loss aversion

The aim of this paper is to use rationality arguments to back out the hedonic MRS from reference-dependent choices under loss aversion. However, one may ask whether it makes sense to appeal to rationality when loss aversion itself may not [\(2005\)](#) on the distinction between hedonic and choice preferences.

be rational. It is therefore of interest to see if loss aversion can be thought to have a rational basis. In this section we show that this is indeed the case. It is possible to formulate a fairly general model of decisions under Knightian uncertainty in which loss aversion is rational. This result is, of course, also of interest in its own right.

We shall make use of the theory of [Gilboa and Schmeidler \(1989\)](#) (GS), who rationalise preferences in terms of utility in situations with uncertainty. Consider a space X of outcomes and a space Y of finite lotteries over X . The subject is supposed to choose between acts that are random variables in Y . That is, an act is a function f from a space S of states of the world into Y .

Subjects are supposed to have a preference relation defined over acts and this preference relation is supposed to satisfy a number of rationality conditions. In particular, subjects are assumed to be averse to uncertainty. GS then shows that the existence of such preferences is equivalent to the existence of a von Neumann Morgenstern utility u defined over Y and a convex and closed set C of probability measures defined over S , such that the preference relation over acts may be rationalised by a utility function over acts f defined by

$$U(f) = \min\left\{\int u(f)dP \mid P \in C\right\}.$$

Adopting the GS axioms, we may then suppose the existence of u and C and use the U as a utility function defined over acts. Note that when C reduces to a single point, i.e. when the probability distribution over S is known, then ordinary expected utility results. When C is larger, i.e. when there are more possible probability distributions over S , then the minimum operator introduces kinks in

the expected utility function U .

We shall specialise this framework to the present context. Let $X = \{(x_1, x_2)\}$ be the space of outcomes. We consider only two goods, one of these may be money and the other may be a non-market good. We shall only consider acts that correspond to a single point in X , which we may identify with the subset of Y that consists only of lotteries with a single certain outcome. We assume a hedonic utility function $u(x_1, x_2)$.

Assume the subject has a reference $r = (r_1, r_2) \in X$ and that this corresponds to a certain act $f_r(s) = (r_1, r_2)$, i.e. an act that leads to the outcome r regardless of the state of the world.

We shall consider valuation studies where subjects are presented with alternatives defined in terms of some change $x = (x_1, x_2)$ to the reference. For example, we may ask subjects to choose between the reference alternative r and $r + x$. In, e.g., a WTP type choice we would then have $x_1 < 0$ and $x_2 > 0$ if the first good is money.

We postulate that subjects associate a degree of uncertainty to the actual outcomes that are related to the change x . We must specify this in terms of the set C of probability measures over the states of nature S , where we take C to describe changes from the reference. The set C incorporates the information that is present in the reference r and the alternative $r + x$, so C must depend on these two points in X .

There are a number of natural requirements that one can impose on the dependency of C on r and x . First, it is reasonable to suppose that the uncertainty increases with the size of the change x from the reference. Second, there is no uncertainty at the reference. Third, subjects will only assign positive probabilities

to changes of the same sign as x . That is, in evaluating something that is presented as an improvement in dimension i , they will not assign any likelihood to a deterioration in dimension i and vice versa.

With these assumptions we specify C to be some closed convex set of probability distributions with the union of their support equal to

$$\text{supp}(C) = \cup_{\pi \in C} \text{supp}(\pi) = x_1[e^{-\eta_1^-}, e^{\eta_1^+}] \times x_2[e^{-\eta_2^-}, e^{\eta_2^+}],$$

where all $\eta's > 0$. What is going to matter here is the bounds on $\text{supp}(C)$. The bounds are given as functions of the reference and the alternative so as to incorporate the requirements discussed above. The size of the $\eta's$ quantify the degree of uncertainty away from the reference. Note that they are allowed all to be different. They are however assumed to be constant (in a choice occasion). If uncertainty decreases as a result of experience and learning, then C shrinks towards the point x , such that C consists in the limit of one distribution that places unit mass at this point.

We have some generality, since we do not have to specify C itself but only the bounds on the supports of distributions in C . On the other hand the definition is somewhat specific, since the bounds on the supports of distributions in C are given by explicit formulae.

We use the notation that $x^+ = x$ if $x > 0$ and 0 otherwise, while $x^- = -x$ if $x < 0$ and 0 otherwise. With the present definition of C we may then approximate the reference-dependent choice utility function over acts measured as changes

from a given reference by

$$U(x|r) \simeq u(r) + u_1(r) \left(e^{-\eta_1^-} x_1^+ - e^{\eta_1^+} x_1^- \right) + u_2(r) \left(e^{-\eta_2^-} x_2^+ - e^{\eta_2^+} x_2^- \right). \quad (1)$$

Given a reference, this is just linear utility with kinks.² Importantly, this provides an explicit link from hedonic to choice utility.

3 Loss aversion and four valuation measures

We now consider the four valuation measures mentioned in the introduction. Consider again a reference-dependent choice utility function defined over two goods. Denote this again by $U(x|r)$, where $x + r$ is the bundle under evaluation and r is the reference. This may be taken to be the reference-dependent utility of the previous section or we may alternatively just assume a reference-dependent choice utility as a starting point. Choice utility may have a kink at the reference. We therefore allow the left and right derivatives of $U(x|r)$ to be different, using the notation $U_i^+(x|r)$ for a partial derivative from the right and $U_i^-(x|r)$ for a partial derivative from the left. Note that loss aversion is just equivalent to $U_i^+ = U_i^+(0|r) < U_i^-(0|r) = U_i^-$. Consider then small changes x from the reference and differentiate $U(x|r)$ to obtain a linear approximation to utility at the reference

$$U(x|r) \simeq U(0|r) + U_1^+(0|r)x_1^+ - U_1^-(0|r)x_1^- + U_2^+(0|r)x_2^+ - U_2^-(0|r)x_2^-. \quad (2)$$

²This formulation is very similar to the formulation of reference-dependent utility in, e.g., Proposition 2 in [Kőszegi and Rabin \(2006\)](#).

Note that this is exactly the same as was found in (1) in the previous section.

Now consider the four standard trade-offs used in economic valuation studies (Bateman et al., 1997). Define the willingness-to-pay (WTP) as how much the individual is willing to pay in terms of x_1 for a marginal increase in x_2 , relative to the reference. Similarly, the willingness-to-accept (WTA) measures how much extra of x_1 would compensate for a marginal reduction (relative to the reference) in x_2 . The equivalent gain (EG) measures indifference at the reference between receiving an increase in x_1 or in x_2 . The equivalent loss (EL) is the corresponding measure for losses. Using (2), it is easy to show that the four valuation measures are given by:

$$\begin{aligned} WTP &= \frac{U_2^+}{U_1^-} \quad , \quad WTA = \frac{U_2^-}{U_1^+} \\ EG &= \frac{U_2^+}{U_1^+} \quad , \quad EL = \frac{U_2^-}{U_1^-} \end{aligned}$$

In general, these will all be different. Observe that

$$WTP \cdot WTA = EG \cdot EL. \tag{3}$$

This equality follows just because reference-dependent utility is linear for small changes. The equality holds regardless of the properties of the utility function, as long as it has derivatives from the left and the right in each variable. What matters is that there is a function that generates choices. The equality can then be used to test whether observed choices are consistent with subjects maximising a function.

If measurements of WTP, WTA, EG and EL are made in some experiment and subjects' choices are generated from maximisation of possibly reference-

dependent utility, then (3) will hold. Empirical evidence in [De Borger and Fosgerau \(2008\)](#) shows that (3) does in fact hold with great precision in a large survey dataset from a discrete choice experiment designed to measure these four valuations simultaneously. This evidence then indicates that people in fact do make choices in a way that is consistent with them maximizing a function.³

Since loss aversion in this setting is equivalent to $U_i^+ < U_i^-$, it can easily be shown that loss aversion implies the following relationship between the four valuation measures: $WTP < (EG, EL) < WTA$ ([Bateman et al., 1997](#); [De Borger and Fosgerau, 2008](#)).

4 Symmetric loss aversion and the identification of hedonic trade-offs

We now assume the existence of an underlying reference-free hedonic utility function $u(r)$. This may be justified using the results of section 2. To link reference-dependent choice preferences to the underlying reference-free hedonic utility function we specify marginal reference-dependent utilities, evaluated at the reference, as follows, where S is the sign function:

$$U(x|r) = u(r) + u_1(r)e^{-\eta_1 S(x_1)}x_1 + u_2(r)e^{\gamma - \eta_2 S(x_2)}x_2.$$

³The survey comprised 2131 car drivers who were asked to choose between two alternative trips defined in terms of travel cost and time. Choice situations were designed relative to a recent trip subjects had made. Each subject made choices in 8 such situations: two WTP type choices, two WTA, two EG and two EL. The econometric model allowed for observed and unobserved heterogeneity and errors, and estimated first a parameter for the median in the sample of each of the four valuation measures. They were all very significant and very different. Imposing the restriction given by (3) cost only about half a likelihood unit, indicating that the restriction embodied in (3) agrees extremely well with the data. [Fosgerau et al. \(2006\)](#) present similar evidence on datasets where subjects were recruited from other modes of transport.

This formulation is equivalent to (1) and (2) since we are free to multiply utility by any positive number. It is easy to see that loss aversion is now equivalent to $\eta_i > 0$. The parameter γ generates asymmetry in reference-dependent utility; in the absence of loss aversion, γ measures the difference between the marginal rates of substitution of hedonic and choice preferences. Working out the four valuation measures for this parameterization, we find:

$$\begin{aligned} WTP &= \frac{u_2(r)}{u_1(r)} e^{\gamma - \eta_1 - \eta_2} \quad , \quad WTA = \frac{u_2(r)}{u_1(r)} e^{\gamma + \eta_1 + \eta_2} \\ EG &= \frac{u_2(r)}{u_1(r)} e^{\gamma + \eta_1 - \eta_2} \quad , \quad EL = \frac{u_2(r)}{u_1(r)} e^{\gamma - \eta_1 + \eta_2} \end{aligned} \quad (4)$$

Suppose we have estimates of the four valuation measures. Under what conditions do these estimates allow the identification of the trade-off implied by the underlying reference-free utility function? The answer immediately follows from the above expressions. First note that γ is not identified separately, since all the measures involve the product of the MRS and $\exp(\gamma)$. Noting that $WTP \cdot WTA = EG \cdot EL = \left(\frac{u_2(r)}{u_1(r)}\right)^2 e^{2\gamma}$, we need to know γ in order to identify $\frac{u_2(r)}{u_1(r)}$ from observation of WTP and WTA (or equivalently from EG and EL).

In conclusion, loss aversion may lead to a large difference between willingness to pay and willingness to accept (and a smaller difference between equivalent gain and loss) and the question of which value is most appropriate in applied cost-benefit analysis. But if γ is known then the marginal rate of substitution implied by the underlying hedonic preferences can be recovered from the geometric mean of the estimates of WTA and WTP (or, alternatively, EL and EG). Indeed, we have the following proposition.

Proposition 1 *If the degree of asymmetry γ is known, then the hedonic MRS is identified from WTA and WTP or from EL and EG by*

$$\frac{u_2(r)}{u_1(r)} e^\gamma = \sqrt{WTP \cdot WTA} = \sqrt{EL \cdot EG}.$$

5 A rationality argument for symmetric loss aversion

In this section, we argue that there are good economic reasons for imposing symmetric loss aversion. We start from the [Munro and Sugden \(2003\)](#) concept of a reflexive optimum. Consider reference-dependent choice utility maximization under a budget restriction reflecting given prices p and endowment Y . The set of reflexive optima then consists of all bundles r such that no small deviation x exists that maintains the budget ($px = 0$) and increases choice utility. The set of reflexive optima is defined by

$$X^* = \{r : U(x|r) \leq U(0|r) \forall x : px = 0\}.$$

Now reconsider the parametrization of reference-dependent preferences given above. Take an arbitrary reference point in the set of reflexive optima $r \in X^*$ and let x be a small budgetary neutral ($px = 0$) change from r to $r + x$. First looking at changes that marginally raise dimension 1 while reducing dimension 2 and then vice versa, it is easily shown, since $r \in X^*$ and therefore $U(x|r) \leq U(0|r)$, that

$$\ln \frac{p_1}{p_2} + \gamma - \eta_1 - \eta_2 \leq \ln \frac{u_1(r)}{u_2(r)} \leq \ln \frac{p_1}{p_2} + \gamma + \eta_1 + \eta_2. \quad (5)$$

Since the reference-free optimum is characterized by $\frac{p_1}{p_2} = \frac{u_1}{u_2}$ we have the following proposition.

Proposition 2 *The reference-free hedonic optimum belongs to the set of reflexive optima if and only if*

$$-\eta_1 - \eta_2 \leq \gamma \leq \eta_1 + \eta_2. \quad (6)$$

Using the representation of WTP and WTA in (4), this proposition implies that the hedonic MRS is bounded between the WTP and the WTA.⁴

We now consider what value of γ would be rational given the degrees of loss aversion, η_1 and η_2 . We take the perspective that a value of γ must be chosen independently of the particulars of the choice situation in which it is going to be applied. We therefore consider the expected hedonic utility given a budget y , random prices p , and a random hedonic utility function u . We want to require minimal information about the distribution of these objects, reflecting ignorance about the situation in which γ is to be applied. The implications of ignorance will be important for what we can say about the optimal value of γ .

We assume (6) such that the hedonic optimum is always a reflexive optimum. The task is then to select γ to maximize

$$E(u(x)|px = y, x \in X^*).$$

In other words, we select γ to maximize expected hedonic utility conditional on being in a reflexive optimum. We impose some assumptions to characterise ig-

⁴This result is not surprising, but we are not aware that it has been shown previously in the literature.

norance. Otherwise, we require no knowledge about the hedonic utility function, nor of income or of prices, except for a regularity condition.

Recall (5) which may be written as

$$\gamma - \eta_1 - \eta_2 \leq \ln \frac{u_1 p_2}{u_2 p_1} \leq \gamma + \eta_1 + \eta_2 \quad (7)$$

for any point in the set of reflexive optima. Let $\phi(t|px = y) = P(\ln \frac{u_1 p_2}{u_2 p_1} = t|px = y)$ be the density of $\ln \frac{u_1 p_2}{u_2 p_1}$ conditional on $px = y$. Then t is in the interval given by (7). We assume that reflexive optima on either side of the hedonic optimum are equally likely or

$$\phi(t|px = y) = \phi(-t|px = y).$$

We further assume that ϕ is unimodal, such that $t\phi'(t|px = y) < 0$ for $t \neq 0$.⁵ This assumption implies that a location in the middle of the set of reflexive optima is more likely. Define for convenience

$$g(t) = E \left(u(x)|px = y, \ln \frac{u_1 p_2}{u_2 p_1} = t \right).$$

We interpret ignorance to say that the sign of t does provide information about $u(x)$ which implies that $g(t) = g(-t)$. Without loss of generality we also assume that $g(t) > 0$. Then, referring to (7) we may elaborate the expected utility as

$$E(u(x)|px = y, x \in X^*) = \int_{\gamma - \eta_1 - \eta_2}^{\gamma + \eta_1 + \eta_2} g(t)\phi(t|px = y)dt. \quad (8)$$

⁵Becker (1962) assumes just a uniform distribution in a similar situation.

Assume finally that both goods are normal goods at all prices. This condition is equivalent to requiring that $\frac{u_j}{u_i} u_{ii} < u_{ij}$ for $i \neq j$. The condition is slightly stronger than strict convexity of the indifference curves, which (in two dimensions) is equivalent to assuming $\frac{u_2}{u_1} u_{11} - 2u_{12} + u_{22} \frac{u_1}{u_2} < 0$. From the assumptions made here, Appendix A then proves the following proposition.

Proposition 3 *With the assumptions above, $0 = \operatorname{argmax}_\gamma E(u(x) | px = y, x \in X^*)$.*

This means that the expected hedonic utility from maximizing reference-dependent choice utility in a sequence of trades is maximized only if loss aversion is symmetric.

6 Conclusion

A large gap between the WTP and the WTA has been observed in numerous valuation studies. The gap seems to be irreconcilable with standard Hicksian preferences, so we must ask which valuation measure it is appropriate to use in welfare economic evaluations.

We have described a model of rational behaviour under uncertainty. This model provides an explicit link from underlying hedonic preferences relevant for welfare economic evaluation to reference-dependent choice preferences exhibiting loss aversion. Moreover, loss aversion is able to explain the WTP-WTA gap.

One ingredient is needed in order to back out the hedonic MRS of interest from observations of choices under loss aversion, namely the degree of asymmetry of loss aversion. We show that symmetric loss aversion may be considered rational. In this case, the hedonic MRS may be found as the geometric average of the WTP

and the WTA or equivalently of the EG and the EL.

7 Acknowledgements

We are grateful to seminar participants in Copenhagen, Aarhus and Antwerp for comments. Mogens Fosgerau has received support from the Danish Social Science Research Council.

References

- Bateman, I., Munro, A., Rhodes, B., Starmer, C. and Sugden, R. (1997) A Test of the Theory of Reference-Dependent Preferences *The Quarterly Journal of Economics* **112**(2), 479–505.
- Becker, G. S. (1962) Irrational Behavior and Economic Theory *Journal of Political Economy* **70**(1), 1–13.
- Bernheim, B. D. and Rangel, A. (2007) Behavioral Public Economics: Welfare and Policy Analysis with Nonstandard Decision-Makers in P. Diamond and H. Vartiainen (eds), *Behavioral Economics and Its Applications* Princeton University Press chapter 2, pp. 7–84.
- Beshears, J., J.Choi, J., Laibson, D. and C.Madrian, B. (2007) How Are Preferences Revealed? *Working Paper* (<http://www.economics.harvard.edu/faculty/laibson/files/How%20are%20preferences%20revealed%202007%2002%2019.doc>).

- De Borger, B. and Fosgerau, M. (2008) The trade-off between money and time: a test of the theory of reference-dependent preferences *Journal of Urban Economics* **64**(1), 101–115.
- Fosgerau, M., Hjorth, K. and Lyk-Jensen, S. V. (2006) *The Danish Value of Time Study - Results for Experiment 1* www.dtf.dk.
- Gilboa, I. and Schmeidler, D. (1989) Maxmin expected utility with non-unique prior *Journal of Mathematical Economics* **18**, 141–153.
- Horowitz, J. K. and McConnell, K. E. (2003) Willingness to accept, willingness to pay and the income effect *Journal of Economic Behavior & Organization* **51**(4), 537–545.
- Kahneman, D. and Sugden, R. (2005) Experienced Utility as a Standard of Policy Evaluation *Environmental and Resource Economics* **32**(1), 161–181.
- Kahneman, D. and Tversky, A. (1979) Prospect Theory: An Analysis of Decision under Risk *Econometrica* **47**(2), 263–292.
- Köszegi, B. and Rabin, M. (2006) A model of reference-dependent preferences *The Quarterly Journal of Economics* **121**(4), 1133–1166.
- Munro, A. and Sugden, R. (2003) On the theory of reference-dependent preferences *Journal of Economic Behavior & Organization* **50**, 407–428.
- Tversky, A. and Kahneman, D. (1991) Loss Aversion in Riskless Choice: A Reference-Dependent Model *The Quarterly Journal of Economics* **106**(4), 1039–1061.

A Proof of Proposition 1

This appendix is devoted to the proof of proposition 1. Differentiate (8) with respect to γ to find the first order condition for a maximum.

$$\begin{aligned} \frac{dE(u(x)|px = y, x \in X^*)}{d\gamma} &= g(\gamma + \eta_1 + \eta_2)\phi(\gamma + \eta_1 + \eta_2|px = y) \\ &- g(\gamma - \eta_1 - \eta_2)\phi(\gamma - \eta_1 - \eta_2|px = y) \\ &= 0 \end{aligned}$$

The first order condition is satisfied for $\gamma = 0$. We note that $\gamma - \eta_1 - \eta_2 \leq 0 \leq \gamma + \eta_1 + \eta_2$ by assumption. Differentiation of the first order condition shows that the second derivative of the expected utility $E(u(x)|px = y, x \in X^*)$ is always negative, provided that $g'(t) \leq 0$ for $t \geq 0$. In this case, the expected utility has a unique maximum at $\gamma = 0$. Our task is then reduced to showing that indeed $g'(t) \leq 0$ for $t \geq 0$.

The budget constraint, $px = y$, and the definition of $t = \ln \frac{u_1 p_2}{u_2 p_1}$ together identify a unique x as a function of y, p, t . Write just $x(t) = (x_1(t), x_2(t))$ since y, p are given under the expectation operator. We then consider $u(t) = u(x(t))$. Differentiating, we find

$$\begin{aligned} u'(t) &= u_1 x'_1 + u_2 x'_2 \\ &= u_1 x'_1 - \frac{p_1}{p_2} u_2 x'_1 \\ &= \sqrt{u_1 u_2} \sqrt{\frac{p_1}{p_2}} e^{t/2} x'_1 - \sqrt{u_1 u_2} \sqrt{\frac{p_1}{p_2}} e^{-t/2} x'_1 \\ &= (e^{t/2} - e^{-t/2}) \sqrt{u_1 u_2} \sqrt{\frac{p_1}{p_2}} x'_1, \end{aligned}$$

where the second equality follows from the budget constraint and the third equality follows from the definition of t .

The problem is now to find x'_1 . First differentiate the definition of t to find

$$u_{11}x'_1 + u_{12}x'_2 = \frac{p_1}{p_2} (u_{12}x'_1 + u_{22}x'_2) e^t + \frac{p_1}{p_2} u_2 e^t.$$

We can then solve this equation for x'_1 . Use first the budget constraint to find

$$x'_2 = -\frac{p_1}{p_2} x'_1, \text{ such that}$$

$$\begin{aligned} u_{11}x'_1 - u_{12}\frac{p_1}{p_2}x'_1 &= \frac{p_1}{p_2} \left(u_{12}x'_1 - u_{22}\frac{p_1}{p_2}x'_1 \right) e^t + \frac{p_1}{p_2} u_2 e^t \\ &\Downarrow \\ \frac{p_1}{p_2} u_2 e^t &= x'_1 \left(u_{11} - u_{12}\frac{p_1}{p_2} - u_{12}\frac{p_1}{p_2} e^t + u_{22}\frac{p_1}{p_2}\frac{p_1}{p_2} e^t \right) \\ &\Downarrow \\ x'_1 &= \frac{\frac{p_1}{p_2} u_2 e^t}{u_{11} - u_{12}\frac{p_1}{p_2} - u_{12}\frac{p_1}{p_2} e^t + u_{22}\frac{p_1}{p_2}\frac{p_1}{p_2} e^t} \\ &\Downarrow \\ x'_1 &= \frac{u_2 e^t}{\frac{p_2}{p_1} u_{11} - u_{12} - u_{12} e^t + u_{22} \frac{p_1}{p_2} e^t}. \end{aligned}$$

Inserting this into the expression for $u'(t)$ yields

$$\begin{aligned} u'(t) &= (e^{t/2} - e^{-t/2}) \frac{u_1 u_2}{\frac{p_2}{p_1} u_{11} e^{-t/2} - u_{12} e^{-t/2} - u_{21} e^{t/2} + u_{22} \frac{p_1}{p_2} e^{t/2}} \\ &= (e^{t/2} - e^{-t/2}) \frac{u_1 u_2}{\frac{u_2}{u_1} u_{11} e^{t/2} - u_{12} (e^{-t/2} + e^{t/2}) + u_{22} \frac{u_1}{u_2} e^{-t/2}} \end{aligned}$$

The sign of the first term here follows the sign of t , while the fraction is always negative by the assumption on the admissible hedonic utility functions. Hence $g'(t) = Eu'(t) \leq 0$ for $t \geq 0$ as required.